Spin-down Rate of Pinned Superfluid

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ABSTRACT

The spinning down (up) of a superfluid is associated with a radial motion of its quantized vortices. In the presence of pinning barriers against the motion of the vortices, a spin-down may be still realized through "random unpinning" and "vortex motion," as two physically separate processes, as suggested recently. The spin-down rate of a pinned superfluid is calculated, in this framework, by directly solving the equation of motion applicable to only the unpinned moving vortices, at any given time. The results indicate that the pinned superfluid in the crust of a neutron star may as well spin down at the same steady-state rate as the rest of the star, through random unpinning events, while pinning conditions prevail and the superfluid rotational lag is smaller than the critical lag value.

Subject headings: stars: neutron – hydrodynamics – pulsars

1. Introduction

Spinning down (up) of a superfluid at a given rate is associated with a corresponding rate of outward (inward) radial motion of its quantized vortices. If the vortices are subject to pinning, as is observed in the experiments on superfluid Helium (Hedge & Glaberson 1980; Schwarz 1981; Adams, Cieplak & Glaberson 1985; Zieve & Donev 2000) and also assumed for the superfluid in the crust of a neutron star (pinned to the lattice nuclei) (Tsakadze & Tsakadze 1975; Tsakadze & Tsakadze 1980; Alpar 1987; Tilley & Tilley 1990), a spin-down would require also unpinning of the vortices, in order to become moveable. Unpinning may be realized by the combined effects due to the Magnus effect, quantum tunnelling and/or thermal activation. However, the subsequent *radial* motion of the unpinned vortices (before repinning) is a separate *dynamical* process, subject to their equation of motion, apart from the unpinning process. This is a view different than that adopted in the model of "vortex creep" (Alpar et. al. 1984; Jones 1991b; Epstein, Link & Baym 1992), which envisages the spin-down to occur through quantum tunnelling *between* adjacent pinning sites, at different *radial* distances. A critical discussion of the model of vortex creep, as well as further justification of the presently adopted viewpoint, may be found elsewhere (Jahan-Miri 2005a;Jahan-Miri 2005b). The derivation of the spin-down rate of a superfluid, in presence of random unpinning, as discussed here, aims to pay due attention to the dynamical role of the vortex *radial* motion. That is, vortex radial motion accompanies a transfer of the spin-down (-up) torque between the "container" and the bulk superfluid, which has to be necessarily meditated by the *moving* (not the stationary *pinned*) vortices, as in the absence of any pinning (Sonin 1987; Tilley & Tilley 1990). Nevertheless, there exist uncertainties in the (micro)physics of vortex motion, as opposed to the structure of a vortex lattice, as well as in the theoretical understanding of the pinning/unpinning mechanisms. Such issues are beyond the scope of the present discussion, and are dealt with by making justified assumptions. The predicted general relation here reduces to that reported previously (Jahan-Miri 2005a), as an approximate limiting case. Moreover, the present calculation is based on a direct solution of the equation of motion for the (temporarily unpinned movable) vortices, in contrast to the heuristic arguments used in Jahan-Miri 2005a.

2. The Spin-down Rate

Different aspects of the derivation will be first discussed separately, which will be then put together to infer the spin-down rate of a superfluid, in the presence of random unpinning of the vortices.

2.1. Vortex Dynamics

The total number density $n_{\rm v}$, per unit area, of the vortices (pinned and unpinned, as a whole) in a superfluid rotating at a rate $\Omega_{\rm s}$ is

$$\kappa n_{\rm v} = 2\Omega_{\rm s}, \tag{1}$$

where $\vec{\kappa}$ is the vorticity vector of a vortex line directed along its rotation axis. A given rate $\dot{\Omega}_{\rm s}$ of change of the rotation frequency $\Omega_{\rm s}$ of the superfluid is associated with a (averaged) radial velocity v_r of the vortices:

$$\dot{\Omega}_{\rm s} \;=\; -2\,\frac{\Omega_{\rm s}}{r}\,v_r, \tag{2}$$

where r is the distance from the rotation axis, and $v_r > 0$ is in the outward direction. The vortices move with the local superfluid velocity except when there is an external force acting on them. The vortex equation of motion is given as (Sonin 1987):

$$\vec{F}_{\rm ext} + \vec{F}_{\rm M} = 0 \tag{3}$$

where $\vec{F}_{\rm ext}$ is the external force on a vortex, per unit length, exerted by the environment/container of the superfluid. The kinematic side of the equation is represented by the Magnus term $\vec{F}_{\rm M}$, arising from the gradient of the superfluid kinetic energy, (loosely, referred to as a "force" exerted by the superfluid on the vortices) and is given, per unit length of a vortex, as (Sonin 1987)

$$\vec{F}_{\rm M} \;=\; \rho_{\rm s}\vec{\kappa} \times (\vec{v}_{\rm L} - \vec{v}_{\rm s}), \tag{4}$$

where $\rho_{\rm s}$ is the superfluid density, $\vec{v}_{\rm s}$ is the local velocity of the superfluid, and $\vec{v}_{\rm L}$ is the velocity of the vortex-line. Hence, a *radial* motion of a vortex, associated with a spin-down, has to be accompanied and indeed driven by a corresponding *azimuthal external* force $F_{\rm ext}$ acting on the *moving* vortex, instantaneously; a preliminary fact, however.

2.2. Unpinning

In the presence of vortex pinning, the required unpinning of the vortices may be normally achieved (in the laboratory cases, and also in neutron stars) under the influence of the so-called (radial) Magnus "force" (Adams, Cieplak & Glaberson 1985; Alpar et. al. 1984). Given a rotational lag $\omega \equiv \Omega_{\rm s} - \Omega_{\rm L}$ between the rotation frequencies of the superfluid and the vortices (where $\Omega_{\rm L} = \Omega_{\rm c}$, if vortices are further assumed to be pinned and co-rotating with the container/crust), radially directed pinning forces would be effective, and are balanced out by the corresponding component of the Magnus term $(F_{\rm M})_r = \rho_{\rm s}\kappa r\omega$, where $\omega > 0$ corresponds to an outward directed $(F_{\rm M})_r$ (Eq. 4). A critical lag $\omega_{\rm crit}$ is thus defined as the maximum value of the lag that the available pinning forces can sustain. For larger assumed values of the lag, $\omega \geq \omega_{\rm crit}$, however stationary pinning conditions may not be realized, and the spinning down of the superfluid occurs as in the absence of any pinning, while *all* of the vortices move and are influenced by the existing external forces, instantaneously.

In contrast, when $\omega < \omega_{\rm crit}$, which is the case of interest here, vortices might be still released from their pinning sites, though partially and temporarily, due to other unpinning mechanisms, say, random unpinning through quantum tunnelling &/or thermal activation (Alpar et. al. 1984). Vortices unpin randomly, move to new radial positions under the influence of the external forces, and pin again; the superfluid spins down accordingly. However, the crucial distinction, with the above case of complete unpinning due to the Magnus effect, is that at any given instant, only a *fraction* of the total number of the vortices (ie, the movable unpinned ones) take part in the transmission of an spin-down torque to the superfluid. The number density $n_{\rm m}$ of the instantaneously moving vortices, that should be considered in a calculation of the superfluid spin-down rate, may be written as

$$n_{\rm m} = \xi \; n_{\rm v}, \tag{5}$$

where ξ is the fraction of the statistical population of unpinned vortices, at any given time. The superfluid spin-down rate, under the assumed condition of $|\omega| < \omega_{\rm crit}$ and random unpinning of its vortices, would be therefore regulated by the unpinning probability ξ, being the weight function for the instantaneous number of (unpinned) moving vortices. This is in spite of the fact that the spin frequency of the superfluid would be still determined by the total number density $n_{\rm v}$ of the vortices (pinned and unpinned), as in Eq. 1. A determination of the unpinning probability is subject to theoretical uncertainties, as discussed by various authors (Anderson & Kim 1964; Alpar et. al. 1984; Jones 1991b; Epstein, Link & Baym 1992). For definiteness, and comparison, the same prescription as given in the vortex creep model may be used. An energy barrier $\Delta E = E_{\rm p}(1 - \omega/\omega_{\rm cr})$ is associated with the pinning potential, per unit length, and ξ is given as (Alpar et. al. 1984)

$$\xi = \exp\left(-\Delta E/kT\right) = \exp\,[-\frac{E_{\rm p}}{{\rm k}T}\frac{\omega_{\rm crit} - \omega}{\omega_{\rm crit}}], \tag{6}$$

where $E_{\rm p}$ is the pinning energy, k is the Boltzmann constant, and T is the temperature.

2.3. "External" Forces

A spin-down of the superfluid would require, in addition to the freedom of the vortices to move in the interstitial space, also the presence of *azimuthal external* forces acting on the *unpinned moving* vortices, instantaneously (§ 2.1). This is a fundamental requirement, irrespective of the nature of the unpinning mechanism, and also of the presence or absence of any rotational lag between the superfluid and the vortices. It may be noted that a treatment of the possible "pinning" of the vortices to the local minima of energy in the interstitial medium is not addressed here, and only pinning to the localized sites (array of the nuclei in the crust of a neutron star) is considered, with a single $\omega_{\rm crit}$ associated to each superfluid layer, as customary. The external forces on vortices could, in general, be of a viscous drag or a "static" frictional nature (Adams, Cieplak & Glaberson 1985; Jones 1991a). The latter type, associated with the "pinning" forces should not be however confused with the role of pinning forces on the "stationary" pinned vortices co-rotating with the pinning centers. In order for the pinning forces to act as frictional forces and impart a net torque on the superfluid the vortices should remain unpinned due to the effect of the Magnus effect (

Adams, Cieplak & Glaberson 1985). This requires $|\omega| > \omega_{\rm crit}$ (usually assumed to hold in any given shell of the star, being a stronger condition than the actual requirement for each vortex line to unpin with a minimum relative velocity with respect to the local superfluid), which means there should be no stationary pinning, hence no random unpinning, to start with. Therefore, the static frictional forces are not relevant to the case of interest here, where $|\omega| < \omega_{\rm crit}$ is assumed. On the other hand, the viscous drag force depends on the relative azimuthal velocity $v_{\rm rel}$ between the "container" and the *unpinned* vortices, and also on the associated microscopic velocity-relaxation timescale τ_v of the vortices. The drag force $F_{\rm d}$, per unit length, is given, for the case of free vortices in the absence of pinning, as (Alpar & Sauls 1988)

$$n_{\rm v} F_{\rm d} \;=\; \rho_{\rm c} \frac{v_{\rm rel}}{\tau_v}, \tag{7}$$

where $\rho_{\rm c}$ is the effective density of the "container". For the superfluid in the crust of a neutron star, the permeating electron (and phonon) gas co-rotating with the solid crust exert the drag forces on the vortex cores. The "container" in this case would be the "crust" which includes all the other components of the star, apart from the superfluid in the crust, and consists of the solid lattice, phonons, and the permeating electron gas in the crust, as well as the core of the star which is assumed to be tightly coupled to the solid crust.

2.4. Relative Rotation of the Vortices

In order to determine the azimuthal component of the relative velocity, $v_{\rm rel}$, or equivalently the relative rotational frequency $\Delta\Omega \equiv |\Omega_{\rm m} - \Omega_{\rm c}|$ between the unpinned vortices (rotating at a rate $\Omega_{\rm m}$) and the crust, one might distinguish between two distinct possibilities for the *initial conditions* upon unpinning. When a vortex (segment) becomes unpinned it might be expected to

i) either, *initially tend* to maintain its overall co-rotation with the pinned vortex lattice and the crust , as before unpinning, hence $\Delta\Omega = 0$ *initially upon unpinning* (ie. $\Omega_{\rm m} = \Omega_{\rm L} = \Omega_{\rm c}$, where $\Omega_{\rm L}$, defined earlier, is the rotation rate of the pinned vortex lattice, in contrast to $\Omega_{\rm m}$ for the temporary unpinned moveable vortices). This could arise due to the general requirement for a locally uniform vortex distribution imposed by the minimization of the free energy (Stauffer & Fetter 1968), assuming that the relaxation to the state of minimum energy of the system for the new pinning conditions is achieved quickly enough compared to the other timescales involved.

Thence, if the crust is not itself being acted upon by any external torque, the situation may persist as the steady state, while the superfluid keeps rotating at a different rate than its container (and the vortices), keeping ω constant with time. The unpinned vortices would be however under the influence of a radial Magnus effect $(F_{\rm M})_r$, corresponding to the assumed value of the lag ω (Eqs 3 & 4). The tension of a vortex line might be invoked as a possible source for counter balancing the radial Magnus term, in the vortex equation of motion, for such unpinned vortices having no radial motion.

If, on the other hand, the crust is itself being spun down by an external torque, which is the case for a neutron star, a relative azimuthal velocity could then develop, with the steady-state magnitude (see § 2.5, below)

$$\Delta\Omega \sim \frac{N}{I_{\rm c}}\,\tau_v, \tag{8}$$

where $I_{\rm c}$ is the moment of inertia of the crust (the rest of the star apart from the superfluid part considered), and N is the magnitude of the external torque acting primarily on the crust.

ii) Else, an unpinned vortex might jump to a rotation frequency same as the superfluid, instantaneously upon unpinning. Hence,

$$\Delta\Omega \sim \Omega_{\rm s} - \Omega_{\rm L} \equiv \omega. \tag{9}$$

The supporting argument for such an assumption would be the fact that, in general, vortices are expected to move with the local superfluid velocity; also a general requirement of the vortex dynamics in the absence of external forces on the vortices (§ 2.1). An instantaneous change of the rotational velocity is indeed permitted for the vortices, being massless fluid configurations, in the usual approximation of zero inertial mass for a vortex (Sonin 1987; Baym & Chandler 1983).

Thence, if the crust is not itself being acted upon by any external torque ($N = 0$), the superfluid would be spun down at the expense of spinning *up* of the crust, and ω decreases gradually. In the presence of negative external torque N, however, ω may as well increase with time.

Either of the above two possibilities might provide a better approximation depending on whether a vortex unpins as a whole along its length, or only small segments of it are unpinned randomly. For the superfluid in the crust of a neutron star, simultaneous unpinning of a vortex as a whole must be ruled out, given the huge number of the pinning centers (ie. the nuclei of the solid crust) along each vortex (having a length of a km or so); hence case *(i)* should be more probable. In contrast, case *(ii)* might be the proper choice for the laboratory experiments in which a vortex pins only at its end points (Hedge & Glaberson 1980; Schwarz

1981). Further theoretical work may indicate the extent to which the (statistically averaged) motion of individual vortices could deviate from a uniform local density, and distinguish between the above alternative possibilities for the initial conditions of the rotation rate of the vortices upon unpinning. The relaxation timescale of the vortex array to the new conditions, in each case, would be likewise relevant for making a decision. Also, the distinct behavior of the superfluid spin-down, for $N = 0$, in the two cases might be possible to be tested, experimentally.

2.5. General Two-Component Rotation

An assumed general model of a normal (non-superfluid) component plus the "crust", with moments of inertia $I_{\rm n}$ and $I_{\rm c}$, and rotation frequencies $\Omega_{\rm n}$ and $\Omega_{\rm c}$, respectively, under the influence of an external negative torque $-N$ acting primarily on the crust-component, would obey the following dynamical relations (Baym et. al. 1969)

$$I_{\rm n}\dot{\Omega}_{\rm n} = I_{\rm c}\frac{\Omega_{\rm c}-\Omega_{\rm n}}{\tau_v}, \tag{10}$$

$$I_{\rm c}\dot{\Omega}_{\rm c} = -N - I_{\rm c}\frac{\Omega_{\rm c}-\Omega_{\rm n}}{\tau_v}, \tag{11}$$

where $I = I_{\rm c} + I_{\rm n}$, and τ_v is the velocity-relaxation time for the dissipation of microscopic relative motion between the constituent particles of the two components. A solution of the two coupled equations indicate exponential relaxations of the rotation frequencies $\Omega_{\rm c}(t)$ and $\Omega_{\rm n}(t)$, with time t. The exponential time constant $\tau_{\rm D}$, referred to as the dynamical coupling timescale of the system, is given as

$$\tau_{\rm D} = \frac{I_{\rm n}}{I}\,\tau_v. \tag{12}$$

In the case of a superfluid component, the relation between $\tau_{\rm D}$ and τ_v would be in general different than that in Eq. 12, as discussed below. Further, the steady-state behavior inferred from the asymptotic solutions of Eqs 10 & 11 indicate a relative rotation difference $\Delta\Omega_{\rm ss}$ such that

$$\Delta\Omega_{\rm ss} = \Omega_{\rm n} - \Omega_{\rm c} = -\frac{I}{I_{\rm c}}\,\tau_{\rm D}\,\dot{\Omega}_\infty, \tag{13}$$

where $\dot{\Omega}_\infty = -\frac{N}{I}$ is the steady-state spin-down rate of either component. The latter relation (Eq. 13) is however expressing a general dynamical relation, applicable also to the case of a superfluid component, with the reservation that the relative rotation of the vortices (not the superfluid) and the crust would be the relevant quantity.

2.6. The Superfluid Dynamical Relaxation time

In contrast to the above formulation of a two-component system, the dynamical coupling time scale of a superfluid is associated with the relaxation of its vortices to their new positions, in response to the existing torque on the superfluid. The added complexity is due to the fact that, unlike the particles of a normal component, the relaxation of vortices involves both their azimuthal as well radial displacements. Moreover, in the case of random unpinning a further complication is that only a fraction of the total vortices are effectively moving, at any given time. For a pinned superfluid with a total number density of the vortices $n_{\rm v}$, per unit area, random unpinning events at a rate ξ may result in a statistical population of free potentially movable vortices, with a number density $n_{\rm m} = \xi n_{\rm v}$ (Eq. 5), at any given time while $|\omega| < \omega_{\rm crit}$. Likewise, looking at any given vortex over a large enough time period (larger than the associated pinning/unpinning intervals), it would move and take part in the relaxation process for only a fraction ξ of the time, and spends the rest of it, $(1 - \xi)$ fraction, as stationary pinned and decoupled. The drag force on any unpinned moving vortex is nevertheless the same as in the normal case when all of the vortices are free and mobile, under the same assumed conditions for the scattering processes and relative velocities (same τ_v and $v_{\rm rel}$). Also, the instantaneous kinematic contribution of the vortices in the superfluid spin frequency is the same irrespective of their pinning/unpinning states. Therefore, the equation of motion of each vortex, governing the time behavior of its radial displacement between successive pinning events, would be exactly the same as in the absence of any pinning (Eq. 14, below). A superfluid rotational relaxation would nevertheless be achieved via rearrangement of the (radial) positions of *all* vortices. The distinction between a pinned subgroup and another unpinned is meaningful only for the instantaneous considerations, and not for a long term relaxation process. This would be further justified if the vortices (being indistinguishable fluid entities) are required to maintain a locally uniform density and more so if the time between successive pinning/unpinnings for each vortex (being of the order of the travel time between adjacent pinning sites, which are the atomic nuclei in the solid crust of a neutron star) is much shorter than the associated relaxation time. Thus the vortices, under the assumed pinning conditions, take part in the relaxation process *as a whole*, even though each undergoes an intermittent cycle of movements and halts.

The relaxation time, in the absence of any pinning, is deduced from a solution of the vortex equation of motion (Eq. 3) for the radial $r_{\rm v}(t)$ and azimuthal $\phi_{\rm v}(t)$ components of the vortex position in polar coordinates, as a function of time t (Alpar & Sauls 1988; Jahan-Miri 1998):

$$r_{\rm v}(t) \quad = \quad r_0 \left[\frac{\Omega_{\rm s0}}{\Omega_{\rm c0}} + \left(1 - \frac{\Omega_{\rm s0}}{\Omega_{\rm c0}} \right) e^{-t/\tau_{\rm D}} \right]^{1/2} \tag{14}$$

$$\phi_{\rm v}(t) \;=\; \phi_0 + \Omega_{\rm c0}t + K\ln\left(\frac{r_{\rm v}(t)}{r_0}\right) \tag{15}$$

where 0-subscripts indicate initial values at $t = 0$ corresponding to an assumed departure from an earlier state of co-rotation of the superfluid (vortices) and the crust, and $K = \frac{\rho_{\rm s}\kappa n_{\rm v}}{\rho_{\rm c}}\,\tau_v$. The relaxation time $\tau_{\rm D}$ needed for the simultaneous re-adjustment of the vortices in both radial and azimuthal directions in response to the exiting torque on the superfluid, ie. the dynamical coupling time scale, is given as

$$\tau_{\rm D} \;=\; \frac{K + \frac{1}{K}}{2\Omega_{\rm c}\left(1 + \frac{I_{\rm s}}{I_{\rm c}}\frac{\Omega_{\rm s}}{\Omega_{\rm c}}\right)}, \tag{16}$$

where $\frac{\rho_{\rm s}}{\rho_{\rm c}} = \frac{I_{\rm s}}{I_{\rm c}}$, and $\kappa n_{\rm v} = 2\Omega_{\rm s} \approx 2\Omega_{\rm c}$ have been used, omitting the zero subscripts.

In the case of pining, the radial position of each vortex changes according to the same equation 14, between its successive pinned states, followed by a halt in its motion until unpinning again. For definiteness, we assume the typical time period $t_{\rm c}$ that any given vortex undergoes a pinning/unpinning cycle is much shorter than the sought relaxation time scale of the system. This should be the relevant limit for the case considered, given the microscopic distances between pinning centers which set the the order of magnitude of the typical distance that is travelled by an unpinned vortex before re-pinning. This length scale together with the typical relative (radial as well as azimuthal) velocities of the vortices with respect to the crust will set the period $t_{\rm c}$, for a given unpinning probability ξ. Thus, one needs to do some averaging over successive movements and stationary states of each vortex in order to infer an exponential-like time behavior for its radial displacement, hence deducing a dynamical relaxation time, comparable to the case of no pining. We try three different averaging methods, which nevertheless give consistent results at least for the relevant limiting cases.

2.6.1. dynamical averaging

As indicated, any given vortex is influenced, for a fraction ξ of the time, by the *same* external force $\vec{F}_{\rm ext}$ as if there where no pinning, and zero azimuthal force in the rest of its time. The time averaged motion of the vortex may be thus determined, in the linear approximation, by a time averaged value of the force, $\overline{\vec{F}}$. Assuming further that $\vec{F}_{\rm ext}$ remains constant during the motion of the vortex between its successive pinning states, one derives simply

$$\overline{\vec{F}} \;=\; \xi\,\vec{F}_{\rm ext} \tag{17}$$

$$= \xi \frac{\rho_{\rm c}}{n_{\rm v}} \frac{\vec{v}_{\rm c} - \vec{v}_{\rm L}}{\tau_v}, \tag{18}$$

for the effective value of the external force on each vortex, per unit length, in the presence of pinning. The latter assumption of the constant force is justified since it is being applied to a time period much shorter than the relaxation time scale of the system.

Alternatively, the equation of motion of the vortices (Eq. 3, which applies to a single vortex, per unit length, as such) might as well be integrated and averaged over radial distances (radial shells) much larger than the microscopic distances between the pinning sites, in the crust of a neutron star. This is justified by the general requirement for a uniform local density of the vortices, which supports the validity of a fluid dynamical approach to the superfluid dynamics, in general (Sonin 1987; Baym & Chandler 1983). Given the km size of the superfluid in the present case, the integration volume would be populated by a large number of the unpinned and pinned vortices, at any given time. Hence, for a solution of the equation of motion, of the whole vortices within an integration volume, one might as well think in terms of an statistically averaged drag force. The averaging would obviously give the same result as in Eqs 17-18, for the “effective” value of the external force on each vortex, per unit length, in the presence of pinning. The latter derivation of Eqs 17-18 indeed applies instantaneously, and dismisses with the earlier restriction about the short term constancy of the drag force.

Solving the vortex equation of motion (Eq. 3), with $\overline{\vec{F}}$ replaced for $\vec{F}_{\rm ext}$ therein, the results would be similar to those in Eqs 14–16, except for the timescale $\tau_{\rm D}$ which may be replaced by a corresponding quantity $\tau_{\rm P}$ as the the dynamical coupling time scale of the pinned superfluid:

$$\tau_{\rm P} = \frac{\frac{K}{\xi} + \frac{\xi}{K}}{2\Omega_{\rm c}\left(1 + \frac{I_{\rm s}}{I_{\rm c}}\frac{\Omega_{\rm s}}{\Omega_{\rm c}}\right)} \tag{19}$$

$$\sim \frac{I_{\rm c}}{I}\left[\frac{I_{\rm s}}{I_{\rm c}}\,\frac{\tau_v}{\xi} + \frac{I_{\rm c}}{I_{\rm s}}\,\frac{\xi}{4\Omega_{\rm s}^2\,\tau_v}\right]. \tag{20}$$

In the limit of $\tau_v >> \frac{2\pi}{\Omega_{\rm s}}$, which is probably the appropriate limit for an application to the crust of young neutron stars, this reduces to the approximate form

$$\tau_{\rm P} \sim \frac{\tau_{\rm D}}{\xi} \sim \frac{I_{\rm s}}{I}\,\frac{\tau_v}{\xi}, \tag{21}$$

also in agreement with the general results of the above two-component model (Eq. 12), for $\xi = 1$, as expected in that limit where the effect of the vortex radial displacement may be neglected thence vortex relaxation behaves as normal fluid, approximately.

2.6.2. kinematical averaging

The dynamical relaxation time $\tau_{\rm P}$, in presence of pinning, might be as well deduced from the time behavior of the vortex motion, over many successive pinning/unpinning cycles. We are again assuming $\tau_{\rm P} >> t_{\rm c}$, as argued above. As depicted in Fig. 1a, the radial position of *any given* vortex, in the pinned case, describes an exponential-like rise in the radial-position–time diagram over a period $\xi t_{\rm c}$, as predicted by Eq. 14 initially for the case of no pinning, followed by a flat portion extended for another period of time $(1-\xi)t_{\rm c}$. This pattern would be then repeated, with the cycle time $t_{\rm c}$, until the final position is reached, corresponding to an assumed final frequency of the superfluid. In comparison to $\tau_{\rm D}$ which is defined as the time constant associated with an exponential fit to the curve (ie., the function in Eq. 14) described by each vortex in the case of no pinning, $\tau_{\rm P}$ would be likewise the time constant associated with an exponential fit to the whole curve representing the overall motion of each vortex. As indicated earlier, for a long term and/or steady state consideration as that of deducing a relaxation time scale ($\tau_{\rm P} >> t_{\rm c}$) for a superfluid in presence of pinning, *all* the vortices play the same and equal role; the distinction between pinned and unpinned populations is but an instantaneous fact. Hence, in the linear approximation which makes it also possible to proceed further analytically, simple geometrical considerations then give (Fig. 1b)

$$\tau_{\rm P} \sim \frac{\tau_{\rm D}}{\xi}, \tag{22}$$

also in agreement with the earlier approximate result as in Eq. 21.

2.7. The Rate

The superfluid spin-down rate may be expressed, in its general form, as (compare Eq. 13)

$$\dot{\Omega}_{\rm s} = \frac{I_{\rm c}}{I}\ \frac{\Delta\Omega_{\rm s}}{\tau_{\rm P}}, \tag{23}$$

noticing that $\tau_{\rm P}$ is, by definition, the characteristic time for the relaxation of a difference in the rotation frequency $\Delta\Omega_{\rm s}$ ($\equiv \Omega_{\rm s} - \Omega_{\rm c}$) between the superfluid and the container/crust. In the case of pinning, however $\Delta\Omega_{\rm s}$ has to be replaced by the relevant quantity $\Delta\Omega$, which was defined earlier (§2.4) as the difference in rotation frequency between the *unpinned movable* vortices and the crust ($\Delta\Omega\ \equiv \Omega_{\rm m} - \Omega_{\rm c}$). Obviously, the superfluid relaxation would be sensitive to the relative rotation of the crust with respect to the moveable vortices as the *only* means for transmission of a torque. In other words, $\dot{\Omega}_{\rm s} = 0$ if and only if $\Delta\Omega = 0$. In contrast, a steady state value of the lag between the superfluid and the *pinned* vortices, implies $\Delta\Omega_{\rm s} \neq 0$, even though $\dot{\Omega}_{\rm s} = 0$, in the absence of the external torque on the container/crust.

Substituting in Eq. 23, for $\tau_{\rm P}$ from Eq. 20, and $\Delta\Omega$ (in place of $\Delta\Omega_{\rm s}$) from either Eq. 8 or Eq. 9, the superfluid spin-down rate in presence of random unpinning of the vortices is predicted to be

- **case i)** if unpinned vortices *tend* to co-rotate with the vortex lattice

$$\dot{\Omega}_{\rm s} \;=\; \frac{N\tau_v}{I_{\rm c}}\left[\frac{I_{\rm s}}{I_{\rm c}}\,\frac{\tau_v}{\xi}+\frac{I_{\rm c}}{I_{\rm s}}\,\frac{\xi}{4\Omega_{\rm s}^2\,\tau_v}\right]^{-1} \tag{24}$$

- **case ii)** if unpinned vortices tend to co-rotate with the bulk superfluid

$$\dot{\Omega}_{\rm s} \;=\; \omega\left[\frac{I_{\rm s}}{I_{\rm c}}\,\frac{\tau_v}{\xi}+\frac{I_{\rm c}}{I_{\rm s}}\,\frac{\xi}{4\Omega_{\rm s}^2\,\tau_v}\right]^{-1}. \tag{25}$$

The corresponding average vortex radial velocity (Eq. 2) may be written down as well, using the approximate limiting form of $\tau_{\rm P}$ (Eq. 21, or Eq. 22),

$$v_r \sim \begin{array}{ll} \frac{r}{2\Omega_{\rm s}}\frac{N}{I_{\rm s}}\,\xi & \quad\text{case } (i) \\ \frac{r}{2\Omega_{\rm s}}\frac{I_{\rm c}}{I_{\rm s}}\frac{\omega}{\tau_v}\,\xi & \quad\text{case } (ii). \end{array} \tag{26}$$

The dependence on N, ω, and τ_v, even in these simplified forms of the relation, represents the very dependence of the superfluid spin down rate on the *instantaneous torque exerted on the superfluid by its environment/container*. As a specific manifestation of this dependence, the sign of v_r, that is the sign of the change in the superfluid spin rate, is determined by that of N, or ω, in either cases. As expected (§ 2.4) Eq. 26 also confirms that, in the absence of external torque N *on* the the superfluid *container*, the pinned superfluid may either retain its rate or else come to a state of co-rotation with the container, depending on the two possibilities considered for the rotation rate of the vortices upon unpinning. The uncertainties in the (micro)physics of individual vortex motion, within a vortex lattice, prevent from deciding between the two cases. However, the predicted distinct behaviors, for the case of $N = 0$, might be used in possible laboratory experiments as a clue to distinguish between the two cases. As a further confirmation, Eq. 26 (case ii) reduces, as it should, to the correct form expected in the absence of pinning (Adams, Cieplak & Glaberson 1985; Alpar & Sauls 1988; Jahan-Miri 1998), for the limiting case of $\xi = 1$ corresponding to values of $|\omega| \geq \omega_{\rm crit}$, when the Magnus effect prevents (even temporary) pinning to be realized.

The above prediction (Eqs 24 or 25) for the superfluid spin-down rate, driven by random unpinning events with a given probability ξ, is fundamentally different than the earlier predictions (Alpar et. al. 1984; Jahan-Miri 2005a). The correct dependence on the dynamically relevant quantities N, ω , τ_v, and ξ assures a true and instantaneous dependence of the superfluid spin-down rate $\dot{\Omega}_{\rm s}$ (or equivalently v_r) on the sing and magnitude of the actual torque transmitted between the superfluid and its container/environvemnt (the crust). It may be noted that even though the steady-state magnitude of ω would be set by other dynamically independent quantities, however for a transient post-glitch relaxation which is our prime objective here it is indeed an independent evolving quantity, initially determined by the glitch. The opposite dependence on ξ in the two terms at the right hand side of Eq. 20 (appearing also in Eqs 24 or 25) is interesting, and resembles the similar behavior of the relaxation time τ_v. The new prediction reduces to an earlier reported estimate (Jahan-Miri 2005a), only in the approximate form, as in Eq. 26, for the limiting cases indicated (with a correction for the case i therein).

For a quantitative evaluation of the efficiency of the spinning down of a superfluid through random unpinning of its pinned vortices, an order of magnitude estimate of the maximum spin-down rate predicted by the present model (Eqs 24 or 25) may be given, as applicable to the crust of neutron stars. The spin-down rate indeed depends on the instantaneous number of the unpinned vortices, as determined by the unpinning probability function $\xi(\omega)$. The maximum spin-down rate would be achieved for values of $\xi \sim 1$, corresponding to $\omega \sim \omega_{\rm crit}$. Adopting a set of parameter values applicable to post-glitch relaxations in young neutron stars, such as $r \sim 10^6$ cm, $\Omega_{\rm s} \sim 10^2$ rad s^{-1}, $N/I \sim 10^{-10}$ rad s^{-2}, $I_{\rm s}/I \sim 0.02$, $\omega_{\rm crit} \sim 10^{-2}$ rad s^{-1}, $\tau_v \geq 10$ s, the (averaged) radial velocity of the vortices could be (Eq. 26)

$$v_r \sim \begin{array}{ll} 10^{-4}\ \text{cm s}^{-1} & \quad \text{case } (i) \\ 10^{3}\ \text{cm s}^{-1} & \quad \text{case } (ii), \end{array} \tag{27}$$

corresponding to the superfluid spin-down rates (Eq. 2)

$$\dot{\Omega}_{\rm s} \sim \begin{array}{ll} 10^{-8}\ \text{rad s}^{-2} & \quad \text{case } (i) \\ 10^{-1}\ \text{rad s}^{-2} & \quad \text{case } (ii). \end{array} \tag{28}$$

The parameter values used above are indeed case dependent to a large extent, and also the exact expression for $\tau_{\rm P}$ (Eq. 20) should be used for a more accurate quantitative estimate. The much larger uncertainty lies however in deciding between the two cases indicated. Nevertheless, the predicted maximum rate, even for that of case (i), is seen to be generally (much) larger than the observed spin-down rates of the radio pulsars, by at least one order of magnitude for the Crab and much so in the case of other pulsars. Therefore, the pinned superfluid in the crust may as well spin down at the same steady-state rate as the rest of the star, through random unpinning events with an associated value of $\xi \lesssim 0.1$ or much smaller, maintaining a rotational lag smaller than the critical lag value.

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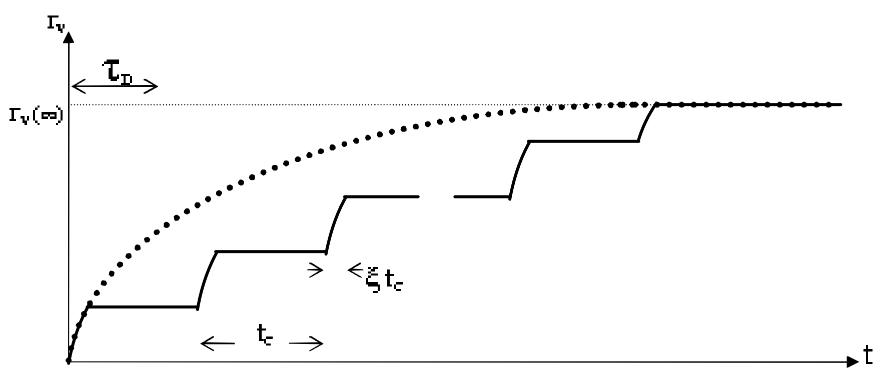
Fig. 1.— **a)** A sketch of the radial displacement, r_v, of any given vortex (in a given shell) versus time, t, during a rotational relaxation of the superfluid. A superfluid relaxation to an assumed final rotation frequency corresponds to a change in the vortex density, hence to certain radial displacement $r_{\rm v}(\infty)$ of each vortex. In the absence of any pinning (*dotted line*) $r_{\rm v}(t)$ is an exponential-like function, as in Eq. 14, with an associated time constant $\tau_{\rm D}$, defined as the dynamical time scale of the superfluid. In presence of pinning with an assumed rate ξ of random unpinning, *all* vortices pin/unpin intermittently with a cycle time $t_{\rm c}$. Each vortex describes the same function $r_{\rm v}(t)$ as in Eq. 14, however for only a time period $\xi\ t_{\rm c}$, then it stops and stays pinned for another time period $(1-\xi)\ t_{\rm c}$. The same cycle of motion/halt is repeated until the final displacement $r_{\rm v}(\infty)$ is reached, as required by the assumed final frequency of the superfluid in a given relaxation. **b)** Same as (a), but in the linear approximation, which also makes it possible to deduce, analytically, a corresponding dynamical time scale $\tau_{\rm P}$ for the pinned superfluid. The *thick solid line* is a convenient linear fit to the linear approximation of the actual time behavior (*thin solid line*) of the displacement of each vortex in the presence of pinning and random unpinning, while the *dotted line* is the linear approximation for the case of no pinning. Note that ξ has been largely exaggerated, as compared to its typical expected values, for the demonstration.

(a)



(b)